

## Finite Element Solution of One-Equation Models of Turbulent Flow

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A description is given of the application of the finite element method to solve two-dimensional, incompressible turbulent flow problems using a one-equation turbulence model. The velocity components and the turbulence kinetic energy are approximated over each element by quadratic interpolation functions, while linear interpolation is used for the pressure. The length scale of the turbulence is specified algebraically. The problem is then reduced to the solution of a set of nonlinear algebraic equations and the method of solution is described. Illustrative examples are fully developed turbulent flow in pipes and the plane turbulent shear layer.

### INTRODUCTION

Over the past decade considerable effort has been directed at the development of computer based numerical models capable of analyzing the large range of turbulent flow problems which are of considerable interest in an engineering context. The advent of the high-speed computer has meant that it has been possible to implement the sophisticated models of turbulence first suggested by Prandtl [1] and Kolmogorov [2] and the finite difference method of solution has been extensively applied to solve the resulting system of differential equations [3, 4]. An alternative technique, using finite elements, has, in general, certain advantages over the finite difference method. These advantages, which include the ease with which irregular geometries and non-uniform meshes can be handled and the imposition of natural boundary conditions, have been demonstrated in potential [5] and laminar [6, 7] flow analyses. As yet, however, little use has been made of the finite element method in the analysis of turbulent flow but it is expected that these advantages will also apply to such problems. The authors [8] have successfully used the method utilizing simple turbulence models, where the time averaged Navier-Stokes equations are used with a turbulent viscosity calculated from an algebraic equation. The present paper describes the utilization of the finite element method for the more sophisticated, so-called, one-equation model of turbulence. This requires the simultaneous solution of the time-averaged Navier-Stokes equations coupled with a differential transport equation for the turbulent kinetic energy. The terms representing generation and dissipation of energy which appear in the kinetic energy equation have no counterpart in the Navier-Stokes equations and require special treatment when the equations are discretized. Mis-

handling of these terms can result in the failure to achieve converged solutions to the resulting set of nonlinear algebraic equations.

A convergent solution technique is evolved which is then used to solve fully developed turbulent flow in pipes and the problem of the plane turbulent shear layer. The results are compared with those obtained experimentally [9] and those obtained using the finite difference method [10].

### GOVERNING EQUATIONS

One method of describing two-dimensional, steady-state isothermal incompressible turbulent flow, ignoring body forces, in a Cartesian coordinate system  $(x, y)$  is to utilize the time-averaged Navier-Stokes equations

$$\begin{aligned} U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \nabla^2 U - \frac{\partial}{\partial x} (\overline{u^2}) - \frac{\partial}{\partial y} (\overline{uv}), \\ U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \nabla^2 V - \frac{\partial}{\partial x} (\overline{uv}) - \frac{\partial}{\partial y} (\overline{v^2}), \end{aligned} \quad (1)$$

together with the continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \quad (2)$$

Here  $U$  and  $V$  are the local time-averaged velocities in the  $x$  and  $y$  directions, respectively,  $u$  and  $v$  are the fluctuational velocity components in these directions,  $\rho$  is the fluid density,  $P$  is the local averaged pressure and  $\nu$  is the laminar kinematic viscosity. The quantities  $-\rho\overline{u^2}$ ,  $\rho\overline{uv}$ ,  $-\rho\overline{v^2}$  are termed the Reynolds stresses and are normally replaced by the expressions

$$\begin{aligned} -\rho\overline{u^2} &= 2\mu_T \frac{\partial U}{\partial x}, \\ -\rho\overline{uv} &= \mu_T \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right), \\ -\rho\overline{v^2} &= 2\mu_T \frac{\partial V}{\partial y}, \end{aligned} \quad (3)$$

where  $\mu_T$  is called the turbulent viscosity.

Equations (1)–(3) form a closed set which can be solved for  $U$ ,  $V$ , and  $P$  provided that an acceptable turbulent viscosity model can be defined. Prandtl [1] and Kolmogorov [2] independently proposed the relationship

$$\nu_T = \mu_T/\rho = lk^{1/2} \quad (4)$$

where  $k$  is the time-averaged turbulence kinetic energy and  $l$  is a typical turbulent length scale. By suitable manipulation of the Navier–Stokes equations it can be shown [11] that  $k$  satisfies an equation of the form

$$\begin{aligned}
 & \text{Convection} && \text{Diffusion} \\
 U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y} &= \frac{\partial}{\partial x} \left[ \left( \nu + \frac{lk^{1/2}}{\sigma_k} \right) \frac{\partial k}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \left( \nu + \frac{lk^{1/2}}{\sigma_k} \right) \frac{\partial k}{\partial y} \right] \\
 & \text{Generation} \\
 & + lk^{1/2} \left[ 2 \left( \frac{\partial U}{\partial x} \right)^2 + 2 \left( \frac{\partial V}{\partial y} \right)^2 + 2 \frac{\partial U}{\partial y} \frac{\partial V}{\partial x} + \left( \frac{\partial V}{\partial x} \right)^2 + \left( \frac{\partial U}{\partial y} \right)^2 \right] \\
 & \text{Dissipation} \\
 & - \frac{C_D k^{3/2}}{l}, \tag{5}
 \end{aligned}$$

where  $\sigma_k$  and  $C_D$  are taken to be constant for fully turbulent flows. In the present paper it is assumed that the distribution of the length scale  $l$  is given by an algebraic expression dependent on the flow under consideration. Equations (1)–(5) then describe the one-equation hydrodynamic models of turbulence as defined by Launder and Spalding [3]. The solution of these equations is required in a closed region  $\Omega$  bounded by a curve  $\Gamma$ . It will be assumed that at all points of  $\Gamma$  either  $U$ ,  $V$ , and  $k$  or their normal derivatives are specified.

### FINITE ELEMENT FORMULATION

The region  $\Omega$  is divided into  $M$  quadrilateral elements  $A^e$  and an extended form of the mixed interpolation technique devised by Hood and Taylor [6, 12] is adopted. This means that eight-noded quadrilateral elements, with associated shape functions  $N^J$ , are used to depict variations in velocity, turbulent kinetic energy, turbulent viscosity, and length scale, while four-noded linear elements, with associated shape functions  $M^J$ , are used for the pressure. The element interpolation functions are then defined by

$$\begin{aligned}
 \hat{U} &= N^J U^J, \\
 \hat{V} &= N^J V^J, \\
 \hat{k} &= N^J k^J, \\
 \hat{v}_T &= N^J v_T^J, \\
 \hat{l} &= N^J l^J, \\
 \hat{P} &= M^J P^J,
 \end{aligned} \tag{6}$$

where the summation convention has been adopted, summation over the appropriate number of indices  $J$  in each case being understood.

Substitution of (6) into Eqs. (1), (2), and (5) and applying the weighted residual method, with weighting functions  $N^J$  for Eqs. (1) and (5) and weighting functions  $M^J$  for eq. (2), produce the standard matrix equation

$$\mathbf{AX} = \mathbf{B}, \quad (7)$$

where the nonlinear terms are incorporated in the matrix  $\mathbf{A}$  and the variables  $\mathbf{X}_K$  associated with the  $K$ th node are

$$\mathbf{X}_K = \begin{pmatrix} U^K \\ P^K \\ V^K \\ k^K \end{pmatrix}. \quad (8)$$

It is apparent that there are many possible forms of constructing the matrix  $\mathbf{A}$ . Previous work on the finite element solution of the laminar Navier-Stokes equations [6] has indicated a suitable positioning in the matrix equation for convection and diffusion terms. There are many more possibilities for the positioning of the generation and dissipation terms appearing in Eq. (5), and tests carried out by the authors have indicated that a satisfactory form which does produce converged solutions is that in which the submatrix  $\mathbf{a}_{KL}$  associated with the  $K$ th node which acts on the  $L$ th nodal variables is as follows:

$$\mathbf{a}_{KL} = \sum_{e=1}^M \iint_{A^e} \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ a_4 & 0 & a_5 & 0 \\ a_6 & a_7 & a_8 & 0 \\ a_9 & 0 & a_{10} & a_{11} \end{bmatrix} dA^e, \quad (9)$$

where

$$a_1 = \alpha_1 + N^I(\nu + \nu_{T^I})\alpha_2 - N^K \frac{\partial N^I}{\partial x} \nu_{T^I} \frac{\partial N^L}{\partial x},$$

$$a_2 = \frac{N^K}{\rho} \frac{\partial M^L}{\partial x},$$

$$a_3 = -N^K \frac{\partial N^I}{\partial y} \nu_{T^I} \frac{\partial N^L}{\partial x},$$

$$a_4 = M^K \frac{\partial N^L}{\partial x},$$

$$a_5 = M^K \frac{\partial N^L}{\partial y},$$

$$a_6 = -N^K \frac{\partial N^I}{\partial x} \nu_{T^I} \frac{\partial N^L}{\partial y},$$

$$a_7 = \frac{N^K}{\rho} \frac{\partial M^L}{\partial y},$$

$$a_8 = \alpha_1 + N^I \nu_T^I \alpha_2 - N^K \frac{\partial N^I}{\partial y} \nu_T^I \frac{\partial N^L}{\partial y},$$

$$a_9 = -N^K N^I \nu_T^I \left[ 2 \frac{\partial N^I}{\partial x} U^I \frac{\partial N^L}{\partial x} - \frac{\partial N^L}{\partial y} \left( \frac{\partial N^I}{\partial y} U^I + \frac{\partial N^I}{\partial x} V^I \right) \right],$$

$$a_{10} = -N^K N^I \nu_T^I \left[ 2 \frac{\partial N^I}{\partial y} V^I \frac{\partial N^L}{\partial y} - \frac{\partial N^L}{\partial x} \left( \frac{\partial N^I}{\partial x} V^I + \frac{\partial N^I}{\partial y} U^I \right) \right],$$

$$a_{11} = \alpha_1 + \left( \frac{N^I \nu_T^I}{\sigma_k} + \nu \right) \alpha_2 + \frac{C_D N^K (N^I k^I)^{1/2}}{N^I l^I} N^L,$$

and

$$\alpha_1 = N^K \left( N^I U^I \frac{\partial N^L}{\partial x} + N^I V^I \frac{\partial N^L}{\partial y} \right),$$

$$\alpha_2 = \frac{\partial N^K}{\partial x} \frac{\partial N^L}{\partial x} + \frac{\partial N^K}{\partial y} \frac{\partial N^L}{\partial y}.$$

The right-hand-side vector  $\mathbf{B}$  contains the prescribed surface gradients of velocity. The  $b_k$ th subvector is given by

$$\mathbf{b}_k = \sum_{e=1}^M \int_{\Gamma^e} \begin{Bmatrix} b_1 \\ 0 \\ b_2 \\ b_3 \end{Bmatrix} d\Gamma^e, \quad (10)$$

where  $d\Gamma^e$  is the elemental length of the appropriate boundary, and

$$b_1 = N^K \left[ \left( \frac{\partial U}{\partial x} \right)^p l_x + \left( \frac{\partial U}{\partial y} \right)^p l_y \right] [N^I \nu_T^I + \nu],$$

$$b_2 = N^K \left[ \left( \frac{\partial V}{\partial x} \right)^p l_x + \left( \frac{\partial V}{\partial y} \right)^p l_y \right] [N^I \nu_T^I + \nu],$$

$$b_3 = N^K \left[ \left( \frac{\partial k}{\partial x} \right)^p l_x + \left( \frac{\partial k}{\partial y} \right)^p l_y \right] \left[ \frac{N^I \nu_T^I}{\sigma_k} + \nu \right],$$

where the superscript  $p$  denotes a prescribed value. On boundaries with prescribed values of  $U$ ,  $V$ , or  $k$ , the corresponding surface integrals need not be evaluated as the nodal values themselves are known and can be applied directly to the matrix, Eq. (7), e.g., if a velocity component at node  $K$  is known, the corresponding component of the  $K$ th momentum equation is deleted and replaced by a 1 on the diagonal and the prescribed value inserted in the right-hand-side vector. If the pressure is known at the  $K$ th node then the continuity equation at the  $K$ th node is deleted and replaced as above.

## SOLUTION TECHNIQUE

The first step in the solution technique is the identification of the boundary conditions to be applied on the velocity, the pressure, and the turbulent kinetic energy. In general, the velocity and pressure boundary values result from the definition of the problem under consideration, while the boundary condition on the turbulence energy is calculated directly from experimental observations on the time-averaged velocity fluctuations in flows of similar character. The length scale model to be employed is determined by the character of the flow under consideration and several standard forms exist for confined [3] and free shear [10] flows.

Apart from the continuity equation, the equations to be solved are nonlinear and their solution requires, of necessity, some form of iteration. The iteration process adopted involved the evaluation of the coefficients of the matrix  $\mathbf{A}$  in Eq. (7) from the values of  $U$ ,  $V$ , and  $k$  at the  $n$ th iteration and the subsequent solution of equation (7) to obtain the  $(n + 1)$ st values of  $U$ ,  $P$ ,  $V$ , and  $k$  was carried out using the frontal solution technique adequately outlined by Hood [15]. This linearization procedure is a natural extension of the work of Hood and Taylor [12] for the solution of the Navier–Stokes equations. They also demonstrated that the laminar equations can be solved by starting the iteration process with initial values of  $U$ ,  $P$ , and  $V$  set to zero. However, for the current problem it has been found that large computer times are required to produce convergence from zero initial conditions, while quicker convergence resulted by using, as initial conditions, the velocity field obtained from the analysis of the same problem using a simpler turbulence model (i.e., the mixing length hypothesis [13]).

The initial distribution of turbulence kinetic energy is then achieved by using a large constant value of the turbulent viscosity in Eq. (4).

The steps in the above process may be summarized as follows:

1. Determine boundary conditions to be applied on  $U$ ,  $P$ ,  $V$ , and  $k$ .
2. Specify a suitable length scale model as a flow dependent function of position.
3. Estimate the initial velocity distribution.
4. Calculate the length scale and determine the distribution of  $k$  using Eq. (4) with a constant turbulent viscosity.
5. Evaluate the matrix  $\mathbf{A}$  using current  $U$ ,  $V$ , and  $k$  values.
6. Solve  $\mathbf{AX} = \mathbf{B}$  for new  $U$ ,  $P$ ,  $V$ , and  $k$  values.
7. Determine the length scale and the distribution of the turbulent viscosity.
8. Repeat from step 5 until prescribed convergence criteria are satisfied.

## ILLUSTRATIVE EXAMPLES

The simplest application of the above formulation is to the analysis of fully developed turbulent flow in a smooth-walled pipe, of diameter  $D$  where both the mean

velocity and the level of turbulence is assumed to remain constant at successive sections along the pipe length. The region considered and boundary conditions applied are shown in Fig. 1. Eight finite elements were used across the pipe section arranged with the widths of successive elements decreasing in the ratio 0.35 as the wall is approached. The length scale was obtained by use of the expression

$$l = C_D^{1/4} l_N, \quad (11)$$

where  $l_N$  is Nikuradse's mixing length profile for fully developed pipe flow [3]. This formula was modified as suggested by Wolfshtein [14] in the laminar sublayer and in the laminar turbulent transition region to account for the proximity of the wall. A value of 0.0916 was used for the constant  $C_D$  as suggested by Launder and Spalding [3].

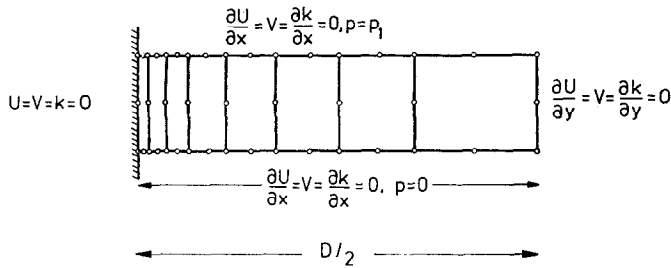


FIG. 1. Region analyzed and boundary conditions for fully developed pipe flow (elements not to scale).

The pressure gradient applied and the laminar viscosity used were adjusted to give a Reynolds number,  $R_N$ , of  $5 \times 10^5$ , to facilitate comparison with the experimental results of Laufer [9]. The comparison between experiment and the results of the finite element model is given in Fig. 2. Figure 2a shows the variation of  $U/U_{CL}$  and  $k^{1/2}/U^*$  with distance from the pipe wall, where  $U_{CL}$  is the center line velocity and  $U^*$  is the shear velocity. The nondimensional near wall velocity distribution is shown in Fig. 2b together with the experimental results and the well-known laws of the wall. It can be seen that, within the limitations of the turbulence model, good agreement has been obtained with experiment.

In the interest of efficiency, this formulation has been used in a slightly modified form to analyze a particular class of free turbulent shear flow problems where the pressure variation is negligible. The particular example presented here is that of the co-flowing mixing layer where two semi-infinite streams interact to produce an ever widening mixing region. The problem is such that, after an initial transition region, the velocities and energy become self-similar in form and the rate of spread of the mixing layer becomes constant. The boundary conditions applied and the finite element mesh used are shown in Fig. 3, and the model of Launder, *et al.* [10], is used for the determination of the length scale variation. In this model the usual assumption

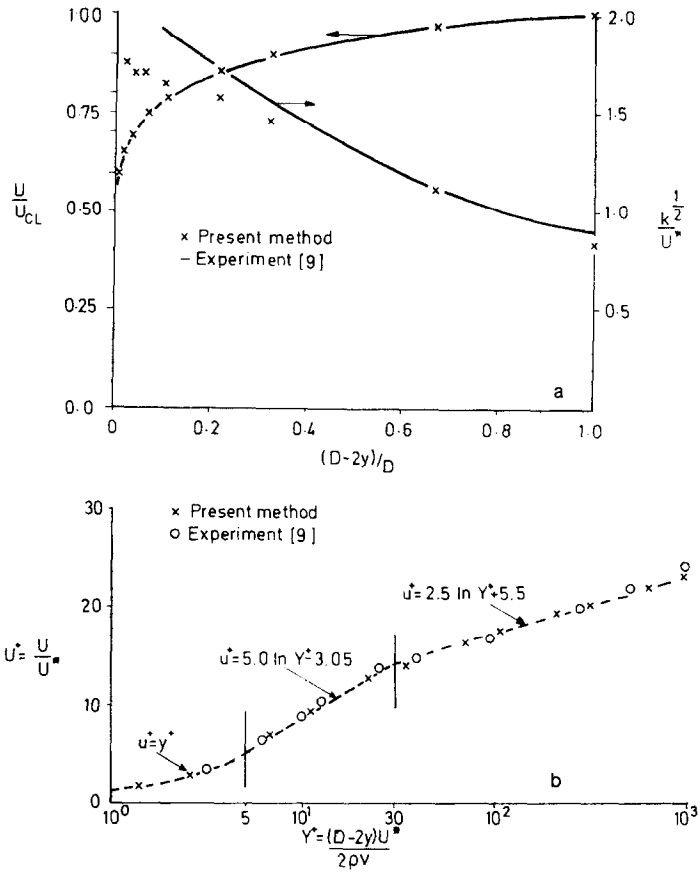


FIG. 2. Fully developed pipe flow,  $R_N = 5 \times 10^5$ . (a) Plot of  $U/U_{CL}$  and  $k^{1/2}/U^*$ . (b) Non-dimensional near wall velocity plot.

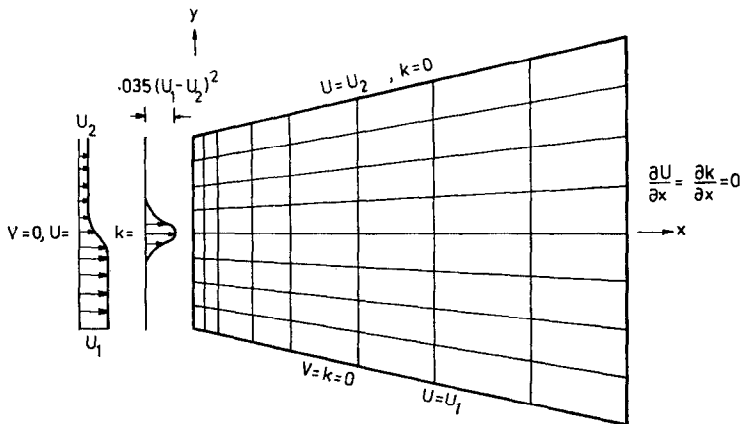


FIG. 3. Solution domain and prescribed boundary conditions for free shear layer.



is made that the length scale at a certain distance downstream is proportional to the width of the mixing layer at that point.

In the self-similar region, the mixing layer is characterized by the spreading parameter,

$$\sigma = \Delta x / \Delta y,$$

where  $\Delta y$  is the increase in width of the mixing layer over a length  $\Delta x$ , and the variation of  $\sigma_0/\sigma$  with different values of the ratio,  $U_2/U_1$ , of the stream velocities is shown in Fig. 4. Here  $\sigma_0$  is the spreading parameter for the case  $U_2 = 0$ . For this example 25 cycles of the iteration scheme were required to produce a 0.5% convergence in the calculated results. These results can be seen to be in excellent agreement with those produced by a finite difference approach [10], which employed forward integration of reduced equations in the streamwise direction.

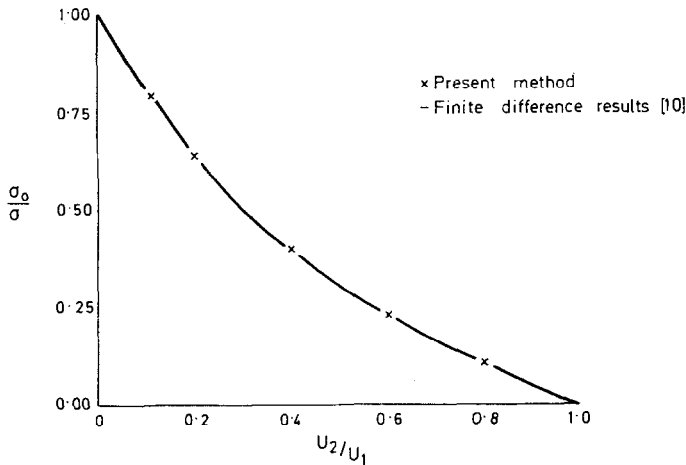


FIG. 4. Variation of rate of spread of mixing region with velocity ratio for free shear layer.

### CONCLUSIONS

A finite element method of solution of two-dimensional incompressible turbulent flows, using a one-equation turbulence model, has been presented and has been shown to be successful in both the analysis of flows of the wall and of free shear turbulence type. Future work will concentrate on the use of the finite element method with more sophisticated turbulence models.

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